

Products of harmonic forms and rational curves

Daniel Huybrechts

Mathematisches Institut

Universität zu Köln

Weyertal 86-90

50931 Köln, Germany

The product of closed forms is closed again. The analogous statement for harmonic forms, however, fails. A priori, there is no reason why the product of harmonic forms should be harmonic again. This phenomenon was recently studied by Merkulov [7]. He shows that it leads to a natural A_∞ -structure on the cohomology of a Kähler manifold. In the context of mirror symmetry Polishchuk made use of (a twisted version of) this A_∞ -structure on elliptic curves to confirm Kontsevich's homological version of mirror symmetry in this case [8].

In this paper we show that this failure of harmonicity in fact happens quite frequently. It usually is related to certain geometric properties of the manifolds and to the existence of rational curves in particular.

Let us briefly indicate the main results for the special case of compact Ricci-flat Kähler manifolds. For a Kähler class $\omega \in H^2(X, \mathbb{R})$ on such a manifold there exists a unique Ricci-flat Kähler form $\tilde{\omega}$ representing it. Let $\mathcal{H}^2(\tilde{\omega})$ denote the space of two-forms harmonic with respect to $\tilde{\omega}$. Of course, for a different Kähler class ω' and the representing Ricci-flat Kähler form $\tilde{\omega}'$ this space might be different.

The main technical result (Prop. 2.3) says that $\mathcal{H}^2(\tilde{\omega})$ is independent of ω if and only if the top exterior power of any harmonic $\alpha \in \mathcal{H}^2(\tilde{\omega})$ is again harmonic. This can be used to interpret the failure of harmonicity of the top exterior power geometrically. Prop. 3.2 asserts that there always exist harmonic two-forms with non-harmonic top exterior power, whenever the Kähler cone (or ample cone) does not form a connected component of the (integral) cone of all classes $\alpha \in H^{1,1}(X, \mathbb{R})$ with $\int_X \alpha^N > 0$.

Note that there are many instances where the Kähler cone is strictly smaller. E.g. this is the case for any Calabi-Yau manifold that is birational to a different Calabi-Yau manifold.

In Sect. 4 we apply the result for K3 surfaces. One finds that on any K3 surfaces containing a rational curve there exists a harmonic two-form α such that α^2 is not harmonic. This can be extended to arbitrary K3 surfaces by using the existence of rational curves on nearby K3 surfaces.

1 Preparations

Let X be a compact Kähler manifold. Then $\mathcal{K}_X \subset H^{1,1}(X, \mathbb{R})$ denotes the Kähler cone, i.e. the open set of all Kähler classes on X . For a class $\alpha \in H^{1,1}(X, \mathbb{R})$ we usually denote by $\tilde{\alpha} \in \mathcal{A}^{1,1}(X)_{\mathbb{R}}$ a closed real $(1,1)$ -form representing α . Let us recall the following version of the Aubin-Calabi-Yau theorem [3]

Theorem 1.1 — *Let X be an N -dimensional compact Kähler manifold with a given volume form $\text{vol} \in \mathcal{A}^{N,N}(X)_{\mathbb{R}}$. For any Kähler class $\omega \in \mathcal{K}_X$ there exists a unique Kähler form $\tilde{\omega} \in \mathcal{A}^{1,1}(X)_{\mathbb{R}}$ representing ω , such that $\tilde{\omega}^N = c \cdot \text{vol}$, with $c \in \mathbb{R}$.*

Since $\tilde{\omega}^N$ is harmonic with respect to $\tilde{\omega}$, this can be equivalently expressed by saying that any Kähler class ω can uniquely be represented by a Kähler form $\tilde{\omega}$ with respect to which the given volume form is harmonic. Note that the constant c can be computed as $c = \int_X \omega^N / \text{vol}(X)$.

Definition 1.2 — *For a given volume form $\text{vol} \in \mathcal{A}^{N,N}(X)_{\mathbb{R}}$ we let $\tilde{\mathcal{K}}_X \subset \mathcal{A}^{1,1}(X)_{\mathbb{R}}$ be the set of Kähler forms $\tilde{\omega}$ with respect to which vol is harmonic.*

By the Aubin-Calabi-Yau theorem the natural projection $\tilde{\mathcal{K}}_X \rightarrow \mathcal{K}_X$ is bijective. But, in general, $\tilde{\mathcal{K}}_X$ is not an open subset of a linear subspace of $\mathcal{A}^{1,1}(X)$ (cf. 2.1). Let $\tilde{\omega} \in \tilde{\mathcal{K}}_X$. The tangent space of $\tilde{\mathcal{K}}_X$ at $\tilde{\omega}$ can be computed as follows. Firstly, we may write $\tilde{\mathcal{K}}_X = \mathbb{R}_+ \times \tilde{\mathcal{K}}_X^c$, where $\tilde{\mathcal{K}}_X^c = \{\tilde{\omega} \in \mathcal{K}_X | \tilde{\omega}^N = c \cdot \text{vol}\}$. Secondly, the infinitesimal deformations of $\tilde{\omega}$ in the direction of $\tilde{\mathcal{K}}_X^c$ are of the form $\tilde{\omega} + \varepsilon \tilde{v}$, where \tilde{v} is a closed real $(1,1)$ -form and such that $(\tilde{\omega} + \varepsilon \tilde{v})^N = \tilde{\omega}^N$. The latter condition gives $\tilde{\omega}^N + \binom{N}{2} \varepsilon \tilde{\omega}^{N-1} \tilde{v} = \tilde{\omega}^N$, i.e. \tilde{v} is primitive. As any closed primitive $(1,1)$ -form is harmonic, this shows that the tangent space of $\tilde{\mathcal{K}}_X^c$ at $\tilde{\omega}$ is the space $\mathcal{H}^{1,1}(\tilde{\omega})_{\mathbb{R}, \text{prim}}$ of real $\tilde{\omega}$ -primitive $\tilde{\omega}$ -harmonic $(1,1)$ -forms. Thirdly, the \mathbb{R}_+ -direction corresponds to the scaling of $\tilde{\omega}$ and this tangent direction is therefore canonically identified with $\mathbb{R}\tilde{\omega}$. Altogether, one obtains that $T_{\tilde{\omega}}\tilde{\mathcal{K}}_X = \mathcal{H}^{1,1}(\tilde{\omega})_{\mathbb{R}}$ is the space of real $\tilde{\omega}$ -harmonic $(1,1)$ -forms. In particular, $\tilde{\mathcal{K}}_X$ is a smooth connected subset of $\mathcal{A}^{1,1}(X)_{\mathbb{R}}$. To make this approach rigorous, one completes $\mathcal{A}^{1,1}(X)$ in the L^2 -topology. The projection of the closed forms to cohomology is a differential map (use e.g. Hodge theory, cf. [4]). The lifted Kähler cone $\tilde{\mathcal{K}}_X$ is the intersection of the space of closed L^2 -forms with the space of sections of the submanifold of the bundle of $(1,1)$ -forms that consists of those positive forms whose top exterior power equals (a scalar multiple of) the given (N,N) -form at every point.

Definition 1.3 — *Let X be a compact Kähler manifold with a given volume form. Then one associates to a given Kähler class $\omega \in \mathcal{K}_X$ the space $\mathcal{H}^{p,q}(\omega) := \mathcal{H}^{p,q}(\tilde{\omega})$ of (p,q) -forms that are harmonic with respect to the unique $\tilde{\omega} \in \tilde{\mathcal{K}}_X$ representing ω .*

Note that two different Kähler forms $\tilde{\omega}_1$ and $\tilde{\omega}_2$ representing the same Kähler class $\omega_1 = \omega_2$ always have different spaces of harmonic $(1,1)$ -forms. Indeed, $\tilde{\omega}_1$ and $\tilde{\omega}_2$ are $\tilde{\omega}_1$ -harmonic respectively $\tilde{\omega}_2$ -harmonic. Since any class, in particular $\omega_1 = \omega_2$, is represented by a unique

harmonic form and $\tilde{\omega}_1 \neq \tilde{\omega}_2$, this yields $\mathcal{H}^{1,1}(\tilde{\omega}_1) \neq \mathcal{H}^{1,1}(\tilde{\omega}_2)$. One might ask more generally what the relation is between the spaces of harmonic forms with respect to different Kähler forms not representing the same Kähler class. It is quite interesting to observe that the dependence of $\mathcal{H}^{1,1}(\tilde{\omega})$ on the Kähler class ω is related to the problem discussed in the introduction. We will try to make this more explicit in the next section.

2 How ‘harmonic’ depends on the Kähler form

Let us begin with the following fact which relates the shape of $\tilde{\mathcal{K}}_X$ to the dependence of $\mathcal{H}^{1,1}(\omega)$ on ω .

Proposition 2.1 — *The subspace $\mathcal{H}^{1,1}(\omega) \subset \mathcal{A}^{1,1}(X)$ is independent of ω if and only if $\tilde{\mathcal{K}}_X$ spans an \mathbb{R} -linear subspace of dimension $h^{1,1}(X)$.*

Proof. Let $\mathcal{H}^{1,1}(\omega) \subset \mathcal{A}^{1,1}(X)$ be independent of $\omega \in \mathcal{K}_X$. Since for any $\omega \in \mathcal{K}_X$ the unique $\tilde{\omega} \in \tilde{\mathcal{K}}_X$ representing it is $\tilde{\omega}$ -harmonic, the assumption immediately yields $\tilde{\mathcal{K}}_X \subset \mathcal{H}^{1,1}(\omega)_{\mathbb{R}}$ for any $\omega \in \mathcal{K}_X$.

Conversely, if $\tilde{\mathcal{K}}_X$ spans an \mathbb{R} -linear subspace of dimension $h^{1,1}(X)$, then this subspace coincides with the tangent space of $\tilde{\mathcal{K}}_X$ at every point $\tilde{\omega} \in \tilde{\mathcal{K}}_X$. But the latter was identified with $\mathcal{H}^{1,1}(\omega)_{\mathbb{R}}$. Hence, the linear subspace equals $\mathcal{H}^{1,1}(\omega)_{\mathbb{R}}$ for any $\omega \in \mathcal{K}_X$ and $\mathcal{H}^{1,1}(\omega)$, therefore, does not depend on ω . \square

Remark 2.2 — The assertion might be rephrased from a slightly different point of view as follows. The bijective map $\tilde{\mathcal{K}}_X \rightarrow \mathcal{K}_X$ can be used to define a differentiable map $\mathcal{K}_X \rightarrow \mathcal{A}^2(X)$ (in the L^2 -topology). The proposition then just says that this map is linear if and only if the Gauss map is constant. It might be instructive to rephrase some of the results later on in this spirit, e.g. Prop. 3.2.

The next proposition states that the ‘global’ change of $\mathcal{H}^{1,1}(\omega)$ for $\omega \in \mathcal{K}_X$ is determined by the ‘harmonic’ behaviour with respect to a single $\omega \in \mathcal{K}_X$.

Proposition 2.3 — *Let X be a compact Kähler manifold of dimension N with a fixed Kähler form $\tilde{\omega}_0$ and volume form $\tilde{\omega}_0^N/N!$. Then the following statements are equivalent:*

- i) The linear subspace $\mathcal{H}^{1,1}(\omega) \subset \mathcal{A}^{1,1}(X)_{\mathbb{R}}$ does not depend on $\omega \in \mathcal{K}_X$.*
- ii) For all $\alpha \in \mathcal{H}^{1,1}(\omega_0)$ one has $\alpha^N \in \mathcal{H}^{N,N}(\omega_0)$.*

Proof. Let us assume *i)*. By the previous lemma the lifted Kähler cone $\tilde{\mathcal{K}}_X$ spans the \mathbb{C} -linear subspace $\mathcal{H}^{1,1}(\omega_0)$. Since $\tilde{\mathcal{K}}_X$ is open in $\mathcal{H}^{1,1}(\omega_0)_{\mathbb{R}}$ and all $\alpha \in \tilde{\mathcal{K}}_X$ satisfy the \mathbb{C} -linear equation

$$\alpha^N = \left(\int_X \alpha^N / \int_X \omega_0^N \right) \cdot \omega_0^N \quad (1)$$

which is an algebraic condition, in fact all $\alpha \in \mathcal{H}^{1,1}(\omega_0)$ satisfy (1). Hence, for all $\alpha \in \mathcal{H}^{1,1}(\omega_0)$ the top exterior power α^N is harmonic, i.e. *ii*) holds true.

Let us now assume *ii*). If $\alpha \in \mathcal{H}^{1,1}(\omega_0)$, such that its cohomology class $\omega := [\alpha]$ is a Kähler class, let $\tilde{\omega} \in \tilde{\mathcal{K}}_X$ denote the distinguished representing Kähler form of ω . If α itself is strictly positive definite, then the unicity of $\tilde{\omega}$ and *ii*) imply $\alpha = \tilde{\omega}$. Thus, the intersection of the closed subset $\mathcal{H}^{1,1}(\omega_0)_{\mathbb{R}}$ with the open cone of strictly positive definite real $(1,1)$ -forms is contained in $\tilde{\mathcal{K}}_X$. This intersection is non-empty, as it contains $\tilde{\omega}_0$. Since $\tilde{\mathcal{K}}_X$ is a closed connected subset of this open cone of the same dimension as $\mathcal{H}^{1,1}(\omega_0)_{\mathbb{R}}$ this yields $\tilde{\mathcal{K}}_X \subset \mathcal{H}^{1,1}(\omega_0)_{\mathbb{R}}$. By Prop. 2.1 one concludes that $\mathcal{H}^{1,1}(\omega)$ does not depend on $\omega \in \mathcal{K}_X$. \square

3 The positive cone

The next proposition is a first step towards a geometric understanding of the failure of harmonicity of α^N for a harmonic form α . To state it we recall the following notation.

Definition 3.1 — *For a compact Kähler manifold X the positive cone $\mathcal{C}_X \subset H^{1,1}(X, \mathbb{R})$ is the connected component of $\{\alpha \in H^{1,1}(X, \mathbb{R}) \mid \int_X \alpha^N > 0\}$ that contains the Kähler cone.*

Note that by definition $\mathcal{K}_X \subset \mathcal{C}_X$.

Proposition 3.2 — *If X is a compact Kähler cone such that \mathcal{K}_X is strictly smaller than \mathcal{C}_X , then for any Kähler form $\tilde{\omega}$ there exists a $\tilde{\omega}$ -harmonic $(1,1)$ -form α such that α^N is not $\tilde{\omega}$ -harmonic.*

Proof. Assume that there exists a Kähler form $\tilde{\omega}_0$ such that for all $\alpha \in \mathcal{H}^{1,1}(\tilde{\omega}_0)$ also α^N is $\tilde{\omega}_0$ -harmonic. We endow X with the volume form $\tilde{\omega}_0^N/N!$. By Prop. 2.3 the lifted Kähler cone $\tilde{\mathcal{K}}_X$ is contained in $\mathcal{H}^{1,1}(\tilde{\omega}_0)$. Since \mathcal{K}_X is strictly smaller than \mathcal{C}_X there exists a sequence $\omega_t \in \mathcal{K}_X$ converging towards a $\omega \in \mathcal{C}_X \setminus \mathcal{K}_X$. As $\tilde{\mathcal{K}}_X$ is contained in the finite-dimensional space $\mathcal{H}^{1,1}(\tilde{\omega}_0)$ the lifted Kähler forms $\tilde{\omega}_t \in \tilde{\mathcal{K}}_X$ will converge towards a form (!) and not only a current $\tilde{\omega} \in \mathcal{H}^{1,1}(\tilde{\omega}_0) \setminus \tilde{\mathcal{K}}_X$. As a limit of strictly positive definite forms $\tilde{\omega}$ is still semi-positive definite. Moreover, $\tilde{\omega}$ is strictly positive definite at $x \in X$ if and only if $\tilde{\omega}^N$ does not vanish at x . By assumption $\tilde{\omega}^N = c\tilde{\omega}_0^N$ with $c = \int_X \omega^N / \int_X \omega_0^N$. Since $\omega \in \mathcal{C}_X$, the scalar factor c is strictly positive. Hence, $\tilde{\omega}^N$ is everywhere non-trivial. Thus $\tilde{\omega}$ is strictly positive definite. This yields the contradiction. \square

The interesting thing here is that the proposition in particular can be used to determine the positivity of a class with positive top exterior power just by studying the space of harmonic forms with respect to a single given, often very special Kähler form:

Corollary 3.3 — *Let X be a compact Kähler manifold with a given Kähler form $\tilde{\omega}_0$. If for all $\tilde{\omega}_0$ -harmonic $(1,1)$ -forms α the top exterior power α^N is also $\tilde{\omega}_0$ -harmonic, then any class $\omega \in \mathcal{C}_X$ is a Kähler class.* \square

We conclude this section with a few examples, where the assumption of the corollary is met *a priori*. In the later sections we will discuss examples where \mathcal{K}_X is strictly smaller than \mathcal{C}_X and where Prop. 3.2 can be used to conclude the ‘failure’ of harmonicity.

Examples 3.4 — *i) If X is a complex torus and ω is a flat Kähler form, then harmonic forms are constant forms and their products are again constant, hence harmonic. In particular, one recovers the fact that on a torus the Kähler cone and the positive cone coincide.*

ii) If for two Kähler manifolds $(X, \tilde{\omega})$ and $(X', \tilde{\omega}')$ with $b_1(X) \cdot b_1(X') = 0$ the top exterior power of any harmonic $(1,1)$ -forms on X or on X' is again harmonic, then the same holds for the product $(X \times X', \tilde{\omega} \times \tilde{\omega}')$. The additional assumption on the Betti-numbers is necessary as the product of two curves shows. Indeed, any $\varphi \in H^{1,0}(X)$, for a curve X , is harmonic, but $\varphi \wedge \bar{\varphi}$ is not. Hence, $\alpha = \varphi \times \bar{\varphi} + \bar{\varphi} \times \varphi$ is a harmonic $(1,1)$ -form on $X \times X'$ with non-harmonic α^2 .

iii) If X is a Kähler manifold, such that $\mathcal{H}^{1,1}(\omega)$ does not depend on ω , then the same holds for any smooth finite quotient of X .

iv) For hermitian symmetric spaces of compact type it is known that the space of harmonic forms equals the space of forms invariant under the real form. As the latter space is invariant under products, the Kähler cone of an irreducible hermitian symmetric space coincides with the positive cone.

4 K3 surfaces

As indicated earlier the behaviour of the Kähler cone is closely related to the geometry of the manifold. We shall study this in more detail for K3 surfaces. The next proposition follows directly from the description of the Kähler cone of a K3 surface.

Proposition 4.1 — *Let X be a K3 surface containing a smooth rational curve. Then for any Kähler form $\tilde{\omega}$ there exists an $\tilde{\omega}$ -harmonic form $(1,1)$ -form α such that α^2 is not harmonic.*

Proof. If X contains a smooth rational curve, then \mathcal{K}_X is strictly smaller than \mathcal{C}_X and we apply Prop. 3.2. Indeed, a smooth rational curve $C \subset X$ determines a (-2) -class $[C]$, whose perpendicular hyperplane $[C]^\perp$ cuts \mathcal{C}_X into two parts and \mathcal{K}_X is contained in the part that is positive on C . \square

If the harmonicity of the top exterior powers fails for a Kähler manifold with a given Kähler form $(X, \tilde{\omega})$ then it should do so for any small deformation of $(X, \tilde{\omega})$. For a Ricci-flat Kähler structure on a K3 surface the argument can be reversed and one can use the existence of

rational curves on arbitrarily near deformations to prove the above proposition on any K3 surface.

Corollary 4.2 — *Let X be an arbitrary K3 surface. If $\tilde{\omega}$ is any hyperkähler form on X , then there exists a $\tilde{\omega}$ -harmonic $(1,1)$ -form α such that α^2 is not $\tilde{\omega}$ -harmonic.*

Proof. Let $H^0(X, \Omega_X^2) = \mathbb{C}\sigma$. Then

$$\begin{aligned}\mathcal{H}^2(\tilde{\omega}) &= \mathcal{H}^{1,1}(\tilde{\omega}) \oplus \mathcal{H}^{2,0}(\tilde{\omega}) \oplus \mathcal{H}^{0,2}(\tilde{\omega}) \\ &= \mathcal{H}^{1,1}(\tilde{\omega}) \oplus \mathbb{C}\sigma \oplus \mathbb{C}\bar{\sigma}\end{aligned}$$

As the space of harmonic forms only depends on the underlying hyperkähler metric g , $\mathcal{H}^{1,1}(\tilde{\omega}) \oplus \mathbb{C}\sigma \oplus \mathbb{C}\bar{\sigma}$ contains $\mathcal{H}^{1,1}(\tilde{\omega}_{aI+bJ+cK})$ for all $(a,b,c) \in S^2$. Here, I, J , and K are the three complex structures associated with the hyperkähler metric g .

Assume α^2 is g -harmonic for all $\alpha \in \mathcal{H}^{1,1}(\tilde{\omega})$. Since $\sigma = \tilde{\omega}_J + i\tilde{\omega}_K$ (up to a scalar factor) and since the product of a harmonic form with the Kähler form is again harmonic, also $\sigma\bar{\sigma}$ is harmonic. This implies that α^2 is harmonic for all $\alpha \in \mathcal{H}^2(\omega)$, as $\sigma^2 = \bar{\sigma}^2 = \alpha\sigma = \alpha\bar{\sigma} = 0$ for $\alpha \in \mathcal{H}^{1,1}(\tilde{\omega})$. Thus, α^2 is g -harmonic for all $\alpha \in \mathcal{H}^{1,1}(\tilde{\omega}_{aI+bJ+cK})$ and all $(a,b,c) \in S^2$. On the other hand, it is well-known that for a non-empty (dense) subset of S^2 the K3 surface $(X, aI+bJ+cK)$ contains a smooth rational curve. Indeed, if $e \in H^2(X, \mathbb{Z})$ is any (-2) -class, then the subset of the moduli space of marked K3 surfaces for which e is of type $(1,1)$ is a hyperplane section. This hyperplane section, necessarily, cuts the complete curve given by the base $\mathbb{P}^1 = S^2$ of the twistor family. Hence, on one of the K3 surfaces $(X, aI+bJ+cK)$ the class e represents a smooth rational curve. Contradiction. \square

Remark 4.3 — What are the bad harmonic $(1,1)$ -forms? Certainly $\tilde{\omega}^2$ is harmonic and for any harmonic form α also $\tilde{\omega}\alpha$ is harmonic. So, if there is any bad harmonic $(1,1)$ -form there must be also one that is $\tilde{\omega}$ -primitive. Most likely, it is even true that the square of any primitive harmonic form is not harmonic. The proof of it should closely follow the arguments in the proof of Prop. 3.2, but there is a slight subtlety concerning the existence of sufficiently many (-2) -classes, that I cannot overcome for the moment. We sketch the rough idea: Assume there exists a $\tilde{\omega}$ -harmonic $\tilde{\omega}$ -primitive real $(1,1)$ -form α such that α^2 is $\tilde{\omega}$ -harmonic. As a $\tilde{\omega}$ -harmonic $\tilde{\omega}$ -primitive $(1,1)$ -form α is also of type $(1,1)$ with respect to any complex structure $\lambda = aI+bJ+cK$ induced by the hyperkähler metric corresponding to $\tilde{\omega}$ (see Prop. 7.5 [5]). Moreover, α is also primitive with respect to all Kähler forms $\tilde{\omega}_\lambda$. Assume that there exists a complex structure $\lambda \in S^2$, such that $\mathcal{C}_X \cap \mathbb{R}[\alpha] \oplus \mathbb{R}\omega_\lambda$ is not contained in \mathcal{K}_X . This condition can be easily rephrased in terms of (-2) -classes and thus becomes a question on the lattice $3U \oplus 2(-E_8)$. It looks rather harmless, but for the time being I do not know a complete proof of it. Under this assumption, we may even assume that in fact $\lambda = I$. Since α^2 is harmonic, in fact β^2 is harmonic for all $\beta \in \mathbb{R}\alpha \oplus \mathbb{R}\tilde{\omega} \subset \mathcal{H}^{1,1}(\omega)$. Going back to the proof of Prop. 3.2, we see that the second part of it can be adapted to this situation and shows that

$\psi^{-1}(\mathcal{K}_X \cap \mathbb{R}[\alpha] \oplus \mathbb{R}\omega) \subset \mathbb{R}\alpha \oplus \mathbb{R}\tilde{\omega}$, where $\psi : \tilde{\mathcal{K}}_X \rightarrow \mathcal{K}_X$. The space $\psi^{-1}(\mathcal{K}_X \cap \mathbb{R}[\alpha] \oplus \mathbb{R}\omega)$ is the space of the distinguished Kähler forms whose classes are linear combinations of $[\alpha]$ and ω . Therefore, all these forms are harmonic and linear combinations of α and $\tilde{\omega}$ themselves. To conclude, we imitate the proof of Prop. 3.2 and choose a sequence $\omega_t \in \mathcal{K}_X \cap \mathbb{R}[\alpha] \oplus \mathbb{R}\omega$ converging towards $\omega' \in \mathcal{C}_X \setminus \mathcal{K}_X$. The corresponding sequence $\tilde{\omega}_t \in \tilde{\mathcal{K}}_X$ is contained in $\mathbb{R}\alpha \oplus \mathbb{R}\tilde{\omega}$ and converges towards a form(!) $\tilde{\omega}'$. As in the proof of Prop. 3.2 this leads to a contradiction.

5 Hyperkähler manifolds

We will try to improve upon Prop. 3.2 in the case of hyperkähler manifolds. In particular, we will replace the question whether the top exterior power α^N of an harmonic form α is harmonic by the corresponding question for the square of α . The motivation for doing so stems from the general philosophy that hyperkähler manifolds should be treated in almost complete analogy to K3 surfaces and that instead of the top intersection pairing one should consider the Beauville-Bogomolov [2] form as the higher dimensional analogue of the intersection pairing for K3 surfaces.

Let us begin by recalling some notations and basic facts. By a compact hyperkähler manifold X we understand a simply-connected compact Kähler manifold, such that $H^0(X, \Omega^2) = \mathbb{C}\sigma$, where σ is an everywhere non-degenerate holomorphic two-form. A Ricci-flat Kähler form $\tilde{\omega}$ turns out to be a hyperkähler form (cf. [2]), i.e. there exists a metric g and three complex structures I, J , and $K := IJ$, such that the corresponding Kähler forms $\tilde{\omega}_{aI+bJ+cK}$ are closed for all $(a, b, c) \in S^2$, such that I is the complex structure defining X , and such that $\tilde{\omega} = \tilde{\omega}_I$. One may renormalize σ , such that $\sigma = \tilde{\omega}_J + i\tilde{\omega}_K$. In particular, multiplying with σ maps harmonic forms to harmonic forms, for this holds true for the Kähler forms $\tilde{\omega}_J$ and $\tilde{\omega}_K$. The positive cone $\mathcal{C}_X \subset H^{1,1}(X, \mathbb{R})$ is a connected component of $\{\alpha \in H^{1,1}(X, \mathbb{R}) \mid q_X(\alpha) > 0\}$, where q_X is the Beauville-Bogomolov form (cf. [2, 5]).

Proposition 5.1 — *Let X be a $2n$ -dimensional compact hyperkähler manifold with a fixed hyperkähler form ω_0 and the unique holomorphic two-form σ . Then $\alpha^2(\sigma\bar{\sigma})^{n-1}$ is harmonic for all $\alpha \in \mathcal{H}^{1,1}(\omega_0)$ if and only if the linear subspace $\mathcal{H}^{1,1}(\omega) \subset \mathcal{A}^{1,1}(X)$ does not depend on $\omega \in \mathcal{K}_X$.*

Proof. Assume that for all $\alpha \in \mathcal{H}^{1,1}(\omega_0)$ also $\alpha^2(\sigma\bar{\sigma})^{n-1}$ is harmonic. If α is in addition strictly positive definite and $\tilde{\omega} \in \tilde{\mathcal{K}}_X$ with $[\alpha] = \omega$, then $\alpha^2(\sigma\bar{\sigma})^{n-1} = \tilde{\omega}^2(\sigma\bar{\sigma})^{n-1}$. We adapt Calabi's classical argument to deduce that in this case $\alpha = \tilde{\omega}$: If $\alpha^2(\sigma\bar{\sigma})^{n-1} = \tilde{\omega}^2(\sigma\bar{\sigma})^{n-1}$, then $(\alpha - \tilde{\omega})(\alpha + \tilde{\omega})(\sigma\bar{\sigma})^{n-1} = 0$. Since α and $\tilde{\omega}$ are strictly positive definite, also $(\alpha + \tilde{\omega})$ is strictly positive definite. By Lemma 6.1 of [5] also $(\alpha + \tilde{\omega})(\sigma\bar{\sigma})^{n-1}$ is strictly positive. As $[\alpha] = \omega = [\tilde{\omega}]$, the difference $\alpha - \tilde{\omega}$ can be written as $dd^c\varphi$ for some real function φ . But by the maximum principle the equation $(\alpha + \tilde{\omega})(\sigma\bar{\sigma})^{n-1}dd^c\varphi = 0$ implies $\varphi \equiv \text{const.}$ Hence, $\alpha = \tilde{\omega}$.

As in the proof of Prop. 3.2 this shows that the intersection of the closed subset $\mathcal{H}^{1,1}(\omega_0)_{\mathbb{R}}$ with the open cone of strictly positive definite forms in $\mathcal{A}^{1,1}(X)_{\mathbb{R}}$ is contained in $\tilde{\mathcal{K}}_X$ and one concludes that $\tilde{\mathcal{K}}_X \subset \mathcal{H}^{1,1}(\omega_0)_{\mathbb{R}}$.

Hence, \mathcal{K}_X spans a linear subspace of the same dimension and, by Lemma 2.1 this shows that $\mathcal{H}^{1,1}(\omega)$ is independent of $\omega \in \mathcal{K}_X$.

Conversely, let $\mathcal{H}^{1,1}(\omega)$ be independent of $\omega \in \mathcal{K}_X$. Then $\tilde{\mathcal{K}}_X \subset \mathcal{H}^{1,1}(\omega)_{\mathbb{R}}$ for any $\omega \in \mathcal{K}_X$. Therefore, $\alpha^2(\sigma\bar{\sigma})^{n-1} = c(\sigma\bar{\sigma})^n$ with $c \in \mathbb{R}$ for α in the Zariski-dense open subset $\tilde{\mathcal{K}}_X \subset \mathcal{H}^{1,1}(\omega)_{\mathbb{R}}$. Hence, $\alpha^2(\sigma\bar{\sigma})^{n-1}$ is harmonic for any $\alpha \in \mathcal{H}^{1,1}(\omega)$ (cf. proof of Prop. 2.3). \square

Of course, as for K3 surfaces one expects that $\mathcal{H}^{1,1}(\omega)$ does in fact depend on ω . This would again follow from the existence of rational curves in every nearby hyperkähler manifold. But it would actually be more interesting to reverse the argument: Assume that X is a hyperkähler manifold, such that for any small deformation X' of X the Kähler cone $\mathcal{K}_{X'}$ equals $\mathcal{C}_{X'}$. I expect that this is equivalent to saying that $\mathcal{H}^{1,1}(\omega)$ does not depend on ω . If for some other reason than the existence of rational curves as used in the K3 surface case this can be excluded, then one could conclude that there always is a nearby deformation X' for which $\mathcal{K}_{X'}$ is strictly smaller than $\mathcal{C}_{X'}$. The latter is expected to imply the existence of rational curves on X' . Along these lines one could try to attack the Kobayashi conjecture, as the existence of rational curves on nearby deformations would say that X itself cannot be hyperbolic. Unfortunately, I cannot carry this through even for K3 surface.

6 Various other examples

Here we collect a few examples where algebraic geometry predicts the failure of harmonicity of the top exterior power of harmonic two-forms. In all examples this is linked to the existence of rational curves.

Varieties of general type. Let X be a non-minimal smooth variety of general type. As I learned from Keiji Ogus this immediately implies that the Kähler cone is strictly smaller than the positive cone. His proof goes as follows: By definition the canonical divisor K_X is big and by the Kodaira Lemma (cf. [6]) it can therefore be written as the sum $K_X = H + E$ of an ample divisor H and an effective divisor E (with rational coefficients). Consider the segment $H_t := H + tE$ with $t \in [0, 1)$. If all H_t were contained in the positive cone \mathcal{C}_X , then K_X would be in the closure of \mathcal{C}_X . If the Kähler cone coincided with the positive cone \mathcal{C}_X , then K_X would be nef, contradicting the hypothesis that X is not minimal. Hence $t_0 := \sup\{t | H_t \in \mathcal{C}_X\} \in (0, 1)$. If H_{t_0} is not nef, then \mathcal{K}_X is strictly smaller than \mathcal{C}_X . Thus, it suffices to show that H_{t_0} is not nef. If H_{t_0} were nef then all expressions of the form $H_{t_0}^{N-i} \cdot H^{i-1} \cdot E$ would be non-negative. Then $0 = H_{t_0}^N = H_{t_0}^{N-1}(H + t_0 E) = H_{t_0}^{N-1} \cdot H + t_0 H_{t_0}^{N-1} \cdot E$, so both summands must vanish. In particular, $0 = H_{t_0}^{N-1} \cdot H = H^2 \cdot H_{t_0}^{N-2} + t_0 H \cdot H_{t_0}^{N-2} \cdot E$. Again this yields the vanishing of both terms and in particular $0 = H^2 \cdot H_{t_0}^{N-2}$. By induction we eventually

obtain $0 = H^{N-1}.H_{t_0}$ and, furthermore, $0 = H^{N-1}.H_{t_0} = H^N + t_0 H^{N-1}.E$. But this time $H^N > 0$ yields the contradiction. Therefore, for a non-minimal variety of general type one has $\mathcal{K}_X \neq \mathcal{C}_X$ and hence there exist harmonic (with respect to an arbitrary Kähler metric) two-forms with non-harmonic top exterior power. Note that a non-minimal variety contains rational curves. As the reader will notice, the above proof goes through on an arbitrary manifold X that admits a big, but not nef line bundle L (replacing the canonical divisor). Also in this case the positive cone and the Kähler cone differ.

Birational Calabi-Yau. Let X and X' be birational Calabi-Yau manifolds, i.e. K_X and $K_{X'}$ are trivial, then the birational map extends to an isomorphism or there exist harmonic $(1,1)$ -forms on X , such that their top exterior power is not harmonic. Again, a non-trivial birational correspondence produces rational curves. As one expects for hyperkähler manifolds that $\mathcal{H}^{1,1}(\omega)$ does depend on the hyperkähler form even when X does not contain a rational curve, e.g. for K3 surfaces, it would be interesting to see an example of a simply-connected Calabi-Yau manifold (in particular not a torus), where it does not.

The same argument could be applied to the case of different birational minimal models (minimal models are not unique!). This shows that in the previous example the Kähler cone could be strictly smaller than the positive cone, even when K_X is nef or ample.

Blow-ups. This example is very much in the spirit of the previous two. Let X be a non-trivial blow-up of a projective variety Y . Then \mathcal{K}_X is strictly smaller than \mathcal{C}_X and, therefore, for any Kähler structure on X there exist harmonic $(1,1)$ -forms with non-harmonic maximal exterior power. Indeed, if L is an ample line bundle on Y then $f^*(L)$ is nef, but not ample, and it is contained in the positive cone. Hence, $f^*(L) \in \mathcal{C}_X \setminus \mathcal{K}_X$. Note that also the first example could be proved along these lines. By evoking the contraction theorem one shows that any non-minimal projective variety X admits a non-trivial contraction to a projective variety Y . The above argument then yields that \mathcal{K}_X and \mathcal{C}_X are different.

7 Chern forms

Let X be a compact Kähler manifold with a Ricci-flat Kähler form $\tilde{\omega}$. If F denotes the curvature of the Levi-Cevita connection ∇ , then the Bianchi identity reads $\nabla F = 0$. The Kähler-Einstein condition implies $\Lambda_{\omega} F = 0$. The last equation can be expressed by saying that F is $\tilde{\omega}$ -primitive. Analogously to the fact that any closed primitive $(1,1)$ -form is in fact harmonic, one has that for F with $\nabla F = 0$ the primitivity condition $\Lambda_{\tilde{\omega}} F = 0$ is equivalent to the harmonicity condition $\nabla * F = 0$. As for untwisted harmonic $(1,1)$ -forms one might ask for the harmonicity of the product F^m . Slightly less ambitious, one could ask whether the trace of this expression, an honest differential form, is harmonic. This trace is, in fact, a scalar multiple of the Chern character $ch_m(X, \tilde{\omega})$.

Question. — Let $(X, \tilde{\omega})$ be a Ricci-flat Kähler manifold. Are the Chern forms $ch_m(X, \tilde{\omega})$ harmonic with respect to $\tilde{\omega}$?

By what was said about K3 surface we shall expect a negative answer to this question at least in this case:

Problem. — Let X be a K3 surface with a hyperkähler form $\tilde{\omega}$. Let $c_2 \in A^2(X)$ be the associated Chern form. Show that c_2 is not harmonic with respect to $\tilde{\omega}$!

So, this should be seen in analogy to the fact that α^2 is not harmonic for any primitive harmonic $(1,1)$ -form α . Here, α is replaced by the curvatur F and α^2 by $tr F^2$. It is likely that the non-harmonicity of c_2 can be shown by standard methods in differential geometry, in particular by using the fact that c_2 is essentially $\|F\| \cdot \tilde{\omega}^2$ (see [3]), but I do not know how to do this.

Furthermore, it is not clear to me what the relation between the above question and the one treated in the previous sections is. I could imagine that the non-harmonicity of ch_m in fact implies the existence of harmonic $(1,1)$ -forms with non-harmonic top exterior power.

Acknowledgements. I wish to thank U. Semmelmann for his interest in this work and M. Lehn and D. Kaledin for making valuable comments on a first version of it. I am most grateful to Keiji Ogiso for its enthusiastic help with several arguments.

References

- [1] *Géométrie des surfaces K3: modules et périodes*. Séminaires Palaiseau. ed A. Beauville, J.-P. Bourguignon, M. Demazure. Astérisque 126 (1985).
- [2] Beauville, A. *Variétés Kählériennes dont la première classe de Chern est nulle*. J. Diff. Geom. 18 (1983), 755-782.
- [3] Besse, A. *Einstein Manifolds*. Erg. 3. Folge Band 10 (1987).
- [4] Demailly, J.-P. *Théorie de Hodge des variétés kählériennes complètes*. in Introduction à la théorie de Hodge. Panoramas et Synthèses 3 (1996).
- [5] Huybrechts, D. *The Kähler cone of a compact hyperkähler manifold*. Preprint (1999) math.AG/9909109
- [6] Kawamata, Y., Matsuda, K., Matsuki, K. *Introduction to the Minimal Model Problem*. Adv. Stud. Pure Math. 10 (1987), 283-360.
- [7] Merkulov, S. *Strong homotopy algebras of a Kähler manifold*. math/9809172
- [8] Polishchuk, A. *Homological mirror symmetry with higher products*. Preprint (1999) math.AG/9901025